

EIGENVALUES OF REAL SYMMETRIC MATRICES

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ABSTRACT. We present a proof of the existence of real eigenvalues of real symmetric matrices which does not rely on any limit or compactness arguments, but only uses the notions of "sup", "inf".

Let $M_n(\mathbb{R})$ denote the set of all real $n \times n$ -matrices and $\text{Sym}_n(\mathbb{R})$ be the set of all $A \in M_n(\mathbb{R})$ that are symmetric. A key ingredient of the "Spectral Theorem" is the existence of a real eigenvalue of a matrix $A \in \text{Sym}_n(\mathbb{R})$. In some way, this uses limit or compactness arguments in \mathbb{R}^n (e.g., [2, Kap. 6, §2], [3]) or the fact that \mathbb{C} is algebraically closed (e.g., [1, §6.4]). Usually, none of these are available in a first course on linear algebra; in any case, it seems desirable to isolate the bare "analytic" prerequisites of this basic result about matrices. We present here a slight variation of the argument in [2], which refers at only one place to the completeness axiom for \mathbb{R} .

For $v, w \in \mathbb{R}^n$ (column vectors) we let $\langle v, w \rangle := {}^t v \cdot w$ denote the usual scalar product (${}^t v$ is the transpose of v , that is, a row vector). The Euclidean norm of v is denoted by $\|v\| = \sqrt{\langle v, v \rangle}$. We define the norm of a matrix $A = (a_{ij}) \in M_n(\mathbb{R})$ by $|A|_\infty = \max\{|a_{ij}| : 1 \leq i, j \leq n\}$. All we need to know about these norms is the following inequality:

$$(\dagger) \quad \|A \cdot v\| \leq \sqrt{n}^3 |A|_\infty \|v\| \quad \text{for all } v \in \mathbb{R}^n.$$

This easily follows from the inequalities $|w|_\infty \leq \|w\| \leq \sqrt{n}|w|_\infty$ and $|A \cdot w|_\infty \leq n|A|_\infty |w|_\infty$; we set $|w|_\infty = \max\{|w_1|, \dots, |w_n|\}$ for any $w = {}^t(w_1, \dots, w_n) \in \mathbb{R}^n$.

Remark 1. Let $A \in \text{Sym}_n(\mathbb{R})$. By a finite sequence of row and column operations, A can be transformed by congruence into a diagonal matrix. That is, there is a nonsingular real matrix P and a real diagonal matrix D such that $A = {}^t P \cdot D \cdot P$; e.g., [1, §6.7]. Since positive real numbers have square roots, we can further assume that all non-zero diagonal entries of D are ± 1 . Now assume that $A \succeq 0$, that is, $\langle v, A \cdot v \rangle \geq 0$ for all $v \in \mathbb{R}^n$. (Such a matrix is called positive semidefinite.) Then all non-zero diagonal entries of D must be $+1$. Consequently, we have the implication:

$$A \succeq 0 \text{ and } \det(A) \neq 0 \quad \Rightarrow \quad A = {}^t P \cdot P \quad \text{with } P \in M_n(\mathbb{R}) \text{ invertible.}$$

Remark 2. Let $A = (a_{ij}) \in \text{Sym}_n(\mathbb{R})$. If $v \in \mathbb{R}^n$ is such that $\|v\| = 1$, then all components of v have absolute value ≤ 1 and so $|\langle v, A \cdot v \rangle| \leq \sum_{i,j=1}^n |a_{ij}|$. Hence, the set

$$S(A) := \{\langle v, A \cdot v \rangle : v \in \mathbb{R}^n, \|v\| = 1\} \subseteq \mathbb{R}$$

is bounded. In particular, this set has a greatest lower bound $\mu(A) = \inf S(A)$. We have

$$\langle v, A \cdot v \rangle \geq \mu(A) \|v\|^2 \quad \text{for all } v \in \mathbb{R}^n.$$

This inequality is clear if $v = 0$; if $v \neq 0$, then set $w := v/\|v\|$ and note that $\langle w, A \cdot w \rangle \geq \mu(A)$.

By a limit or compactness argument, one can deduce that there exists a vector $v_0 \in \mathbb{R}^n$ such that $\|v_0\| = 1$ and $\langle v_0, A \cdot v_0 \rangle = \mu(A)$. It then follows easily that v_0 is an eigenvector of A with eigenvalue $\mu(A)$ (see [2, Kap. 6, §2, no. 4]). The proof below avoids this line of reasoning.

Theorem 3. *If $A \in \text{Sym}_n(\mathbb{R})$, then $\mu(A)$ is an eigenvalue of A .*

Proof. Let I_n be the identity matrix and set $B := A - \mu(A)I_n \in \text{Sym}_n(\mathbb{R})$. If $\det(B) = 0$, then $\mu(A)$ is an eigenvalue of A . So let us now assume that $\det(B) \neq 0$. We have $\langle v, B \cdot v \rangle = \langle v, A \cdot v \rangle - \mu(A)\|v\|^2$ for all $v \in \mathbb{R}^n$. Remark 2 shows that $B \succeq 0$ and $\mu(B) = \inf S(B) = 0$. Since also $\det(B) \neq 0$, we can write $B = {}^tP \cdot P$, where $P \in M_n(\mathbb{R})$ is invertible (see Remark 1).

Now, for any $v \in \mathbb{R}^n$, we have $\langle v, B \cdot v \rangle = {}^tv \cdot B \cdot v = \|P \cdot v\|^2$. Furthermore, if $\|v\| = 1$, then $1 = \|P^{-1} \cdot (P \cdot v)\| \leq \sqrt{n}^3 |P^{-1}|_\infty \|P \cdot v\|$, using (\dagger) . Thus, $\langle v, B \cdot v \rangle \geq 1/(n^3 |P^{-1}|_\infty^2) > 0$ for all $v \in \mathbb{R}^n$ such that $\|v\| = 1$, contradicting $\inf S(B) = 0$. \square

Remark 4. The argument also works for Hermitian matrices $A \in M_n(\mathbb{C})$. One just has to use the Hermitian product $\langle v, w \rangle = {}^t\bar{v} \cdot w$ for $v, w \in \mathbb{C}^n$, where the bar denotes complex conjugation. If A is Hermitian, then $\langle v, A \cdot v \rangle \in \mathbb{R}$ for all $v \in \mathbb{C}^n$, so we can define $\mu(A) = \inf S(A)$ as above.

REFERENCES

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